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Static and dynamic stability of uniform shear beam-columns under generalized boundary conditions

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Abstract

The stability and dynamic analyses (i.e., the buckling loads, natural frequencies and the corresponding modes of buckling and vibration) of a 2D shear beam-column with generalized boundary conditions (i.e., with rotational restraints and lateral bracings as well as lumped masses at both ends) and subjected to linearly distributed axial load along its span are presented in a classic manner. The two governing equations of dynamic equilibrium, that is, the classical shear-wave equation and the bending moment equation are sufficient to determine the modes of vibration and buckling, and the corresponding natural frequencies and buckling loads, respectively. The proposed model includes the simultaneous effects of shear deformations, translational and rotational inertias of all masses considered, the linearly applied axial load along the span, and the end restraints (rotational and lateral bracings at both ends). These effects are particularly important in members with limited end rotational restraints and lateral bracings. Analytical results indicate that except for members with perfectly clamped ends, the stability and dynamic behavior of shear beams and shear beam columns are governed by the bending moment equation, rather than the second-order differential equation of transverse equilibrium (or shear-wave equation). This equation is formulated in the technical literature by simple applying transverse equilibrium at both ends of the member "ignoring" the bending moment equilibrium equation. This causes erroneous results in the stability and dynamic analyses of such members with supports that are not perfectly clamped. The proposed equations reproduce as special cases: (1) the non-classical vibration modes of shear beam-columns including the inversion of modes of vibration (i.e. higher modes crossing lower modes) in members with soft end conditions, and the phenomena of double frequencies at certain values of beam slenderness (L/r) and (2) the phenomena of tension buckling in shear beam-columns.

1. Introduction

The lateral stability and dynamic behavior of shear beams, shear beam-columns, and shear buildings are of great importance in structural and earthquake engineering. The vibration and seismic responses of shear beams and framed structures modeled as shear buildings have been studied by many researchers and treated extensively in the technical literature (Thomson [1], Blevins [2], Berg [3], Paz [4], Clough and Penzien [5], Chopra [6], among many others) using different methods, mainly matrix analysis and lumped masses.

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Nomenclature

A gross sectional area of the beam-column
A_s effective shear area of the beam-column
C_1, C_2, C_3 and C_4 unknown constants required in the natural modes of vibration
G shear modulus of the material
J_a and J_b Rotational inertia of the concentrated masses at A and B, respectively
$k = \omega\sqrt{\frac{\bar{m}}{GA_s}}$ wave number (according to the shear-beam model developed by Kausel [10])
κ_a and κ_b stiffness of the flexural connections at A and B, respectively (Force $\times$ distance)
L beam-column span
M bending moment
M_a and M_b lumped masses located at A and B, respectively
$\bar{m}$ mass per unit length of the beam-column
P_o compressive axial load applied at the ends of the beam-column
q uniform distributed axial load that includes the self-weight per unit length of the member
r radius of gyration of the beam mass per unit length $\bar{m}$
S_a and S_b stiffness indices of the lateral bracings at ends A and B of the beam-column, respectively
t time
V shear force normal to the initial axis of the beam-column
Q total shear force along the cross section of the deformed beam-column
x coordinate along the centroidal axis of the beam-column
$y(x)$ shape function of lateral deflection of the centroidal line of the beam-column (see Fig. 1)
γ shear distortion
θ maximum rotation of the cross section of the beam-column caused by the rotational of its end connections
ω natural angular frequency of the beam-column
$Y(x, t)$ $y(x)\sin\omega t$
$\Theta(t)$ $\theta\sin\omega t$

The vibration of beams and beam-columns has been studied by many scientists, physicists, mathematicians and engineers. For instance, Kounadis [7,8] and Kounadis and Sophianopoulus [9] carried out research on the dynamics of beams and columns including the effects of shear deformations, attached end masses (positioning, translational and rotational inertias), elastic end restraints, axial compressive forces and inertias. However, there are still some cases like the free-free and pinned-free shear beams (shown by Kausel [10]) that invalidates the classical theory of Bernoulli–Euler and the classical shear wave equation. Classic solutions for the vibration of beam and beam-column based on the Bernoulli–Euler theory (that neglects the combined effects of shear deflections and rotational inertias along the member) violate the principle of conservation of angular momentum even in slender beams. This is particularly critical in free–free and pinned–free beams with low shear stiffness as shown clearly by Kausel [10]. However, analytical studies carried out by Kausel [10] and more recently by Aristizabal-Ochoa [11] indicate that the dynamic behavior of shear beams with generalized support conditions are governed by the moment equation rather than the second-order differential equation of transverse equilibrium (or shear-wave equation that is utilized in most textbooks and in the technical literature by simple applying transverse equilibrium at the two ends of the member). In the technical literature the moment equilibrium equation is totally "ignored" causing erroneous results in the dynamic analyses of beams with low shear stiffness and with supports that are not perfectly clamped. This is particularly critical in the free–free and the pinned-free cases as shown clearly by Kausel [10] invalidating the classical theory of Bernoulli–Euler and the classical shear wave equation. Classic solutions for the vibration of beams and beam-columns based on the Bernoulli–Euler theory (that neglects the combined effects of shear deflections and rotational inertias along the member) also violate the equation of bending moment equilibrium, and consequently the principle of conservation of angular momentum [11].

Therefore, there is a real need for a simple dynamic approach by which the stability and dynamic behavior (critical loads, buckling modes, natural frequencies and free vibration modal shapes) of shear beam-columns with any end support conditions can be determined directly. This model must be more general than the simple shear beam model developed by Kausel [10] and its solutions must be capable of: (1) reproducing the phenomena of double frequencies and inversion of modes in shear beams described by Kausel [10] and (2) determining the buckling loads and modes, natural frequencies and corresponding modal shapes of vibration of shear beam-columns and shear buildings. The main objective of this publication is to derive in a classic manner the equations by which the stability and free vibration analyses of a 2D shear beam-column with generalized support conditions (i.e., with rotational restraints and lateral bracings as well as lumped masses at both ends) and subjected to a linearly distributed axial load along its span can be carried out directly. The proposed model which is an extension of the shear beam model presented by Kausel [10] includes the combined effects of shear deformations, an axially applied load linearly distributed along its span (that could include axial loads applied at its ends plus the self-weight of the member), the translational and rotational inertias of the member's mass, and of the rotational and translational lumped masses located at the ends of the member. Examples are included that show the simplicity and versatility of the proposed equations in the stability and dynamic analyses of shear beams, shear beam-columns, and shear buildings with generalized boundary conditions and subjected to linearly distributed axial load along its span. The solutions of many classical cases of shear beams and shear beam-columns are listed for convenience and easy reference for the common user.

2. Structural dynamic model

2.1. Assumptions

Consider the uniform shear beam-column that connects extremes A and B, as shown in Fig. 1(a). The element is made of the beam-column itself AB, the end flexural connections κ_a and κ_b (whose dimensions are in force-distance/radian), and the lateral springs or bracings S_a and S_b (whose dimensions are in force/distance) at A and B, respectively. Notice that: (1) κ_a and κ_b vary from zero for pinned connections to infinity for fully restrained connections (i.e., perfectly clamped conditions); and (2) S_a and S_b also vary from zero for unbraced end conditions to infinity for totally braced end conditions.

It is assumed that the beam-column AB: (1) is made of a homogenous linear elastic material with shear modulus G; (2) its centroidal axis is a straight line; (3) is loaded along its centroidal axis x with a linearly distributed axial load of magnitude $P_o + qx$ (where $P_o =$ constant axial load applied at both extremes, and $q =$ an uniform distributed axial load that includes the self-weight of the member); (4) its transverse cross section is doubly symmetric (i.e., its centroid coincides with the shear center) with a total area A, an effective

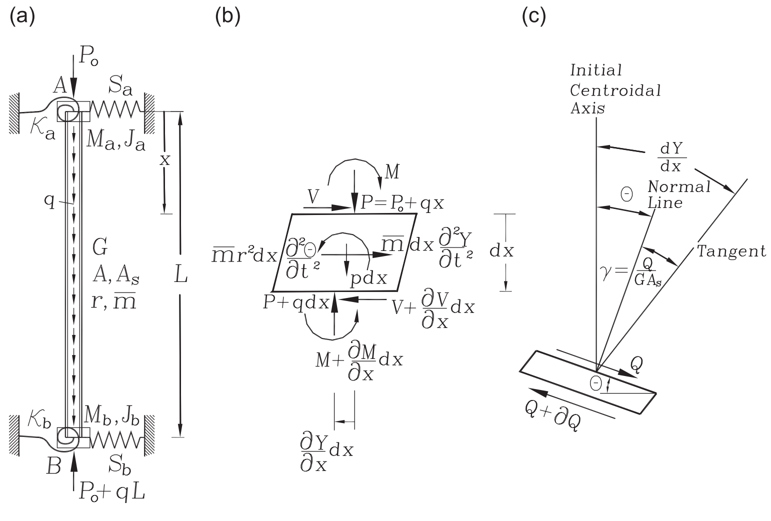


Fig. 1. Structural model: (a) member properties, masses and end connections; (b) forces and moments on the differential element; and (c) rotational and shear deformations at a cross section.

shear area A_s; (5) has a uniform mass per unit length $\bar{m}$, a radius of gyration r, and two lumped masses attached at the extremes A and B of magnitude M_a and M_b and rotational moments of inertia J_a and J_b, respectively; and (6) all transverse deflections, rotations, and strains along the beam are small, so that the principle of superposition is applicable.

2.2. Governing equations

The free-body diagram of the differential element of the member AB located at x from end A, including the effects of translational and rotational inertias subjected to bending, shear and axial loads is shown in Figs. 1(a and b). The corresponding shear distortions γ and rotational deformation Θ as well as the direction of the normal line to the cross section at x and the tangent to the deformed centroidal axis of the element are shown in Fig. 1(c).

Applying transverse and rotational equilibrium to the differential element, Eqs. (1) and (2) are obtained as

$$\frac{\partial V}{\partial x} = \bar{m}\frac{\partial^2 Y}{\partial t^2}, \tag{1}$$

$$\frac{\partial M}{\partial x} = V - \bar{m}r^2\frac{\partial^2 \Theta}{\partial t^2} + (P_o + qx)\frac{\partial Y}{\partial x}. \tag{2}$$

Knowing that according to the "modified shear equation" by Timoshenko and Gere [12, p. 134]: $Q = A_s G\gamma$; $\gamma = (\partial Y/\partial x) - \Theta$; with $Q = V\cos\Theta + (P_o + qx)\sin\Theta \approx V + (P_o + qx)\Theta$ for small angles Θ; and that for shear beam-columns $\partial\Theta/\partial x \cong 0$ and assuming that the values of A_s, r, $\bar{m}$, and G remain constant along the span, the coupled governing Eqs. (1) and (2) become

$$\frac{\partial^2 Y}{\partial x^2} - \frac{\bar{m}}{GA_s}\frac{\partial^2 Y}{\partial t^2} = \frac{q\Theta}{GA_s}, \tag{3}$$

$$M(x) = M_A + \int_0^x \left[V - \bar{m}r^2\frac{\partial^2\Theta}{\partial t^2} + (P_o + qx)\frac{\partial Y}{\partial x}\right] \mathrm{d}x. \tag{4}$$

The solutions to $Y(x, t)$ and $\Theta(t)$ are of the form:

$$Y(x,t) = \left(C_1 \sin kx + C_2 \cos kx + \frac{q\Theta}{GA_s k^2}\right)\sin(\omega t), \tag{5}$$

$$\Theta(t) = \theta\sin(\omega t). \tag{6}$$

where $k = \omega\sqrt{(\bar{m}/GA_s)}$.

C_1, C_2, θ and ω (or k) in Eqs. (5) and (6) are determined from the following four boundary conditions:

$$\text{At } x = 0: \quad V_A = S_a Y(0,t) + M_a\frac{\partial^2 Y(0,t)}{\partial t^2} = (S_a - \omega^2 M_a)Y_a, \tag{7}$$

$$M_A = -\kappa_a\Theta(0,t) - J_a\frac{\partial^2\Theta(0,t)}{\partial t^2} = (-\kappa_a + \omega^2 J_a)\Theta, \tag{8}$$

$$\text{At } x = L: \quad V_B = -S_b Y(L,t) - M_b\frac{\partial^2 Y(L,t)}{\partial t^2} = (-S_b + \omega^2 M_b)Y_b, \tag{9}$$

$$M_B = \kappa_b\Theta(L,t) + J_b\frac{\partial^2\Theta}{\partial t^2}(L,t) = (\kappa_b - \omega^2 J_b)\Theta. \tag{10}$$

The expressions for Y_a and Y_b which can be determined using Eq. (5) when substituted into Eqs. (7) and (9) [taking into account that $V = A_s G\gamma - (P_o + qx)\Theta$, $\gamma = (\partial Y/\partial x) - \Theta$ and $(\partial\Theta/\partial x) \cong 0$], the following two equations in terms of coefficients C_1, C_2, and θ are obtained:

$$kC_1 - \frac{S_a - \omega^2 M_a}{GA_s} C_2 = \left[1 + \frac{P_o}{GA_s} + \frac{q}{k^2 GA_s}\frac{S_a - \omega^2 M_a}{GA_s}\right]\theta, \tag{11}$$

$$\left(k \cos kL + \frac{S_b - \omega^2 M_b}{GA_s} \sin kL\right) C_1 - \left(k \sin kL - \frac{S_b - \omega^2 M_b}{GA_s} \cos kL\right) C_2$$
$$= \left[1 + \frac{P_o + qL}{GA_s} - \frac{qL^2}{(kL)^2 GA_s}\frac{S_b - \omega^2 M_b}{GA_s}\right]\theta. \tag{12}$$

From Eqs. (11) and (12) and taking into account that $\omega^2 = k^2 GA_s/\bar{m}$, C_1 and C_2 can be expressed as follows:

$$C_1 = \frac{\alpha\xi}{k\beta}\theta \tag{13}$$

and

$$C_2 = \frac{\xi}{k\beta}\theta, \tag{14}$$

where

$$\alpha = \frac{\left(\frac{\eta S_a + \xi S_b \cos kL}{GA_s/L} - \xi kL \sin kL\right)\bar{m}L - (\eta M_a + \xi M_b \cos kL)(kL)^2}{\left(\eta kL - \xi kL \cos kL - \frac{\xi S_b}{GA_s/L} \sin kL\right)\bar{m}L + \xi M_b (kL)^2 \sin kL}, \tag{15}$$

$$\beta = \alpha + \frac{M_a}{\bar{m}L} kL - \frac{S_a}{(kL) GA_s/L}, \tag{16}$$

$$\xi = 1 + \frac{P_o}{GA_s} + \frac{qL}{GA_s}\left(\frac{S_a}{(kL)^2 GA_s/L} - \frac{M_a}{\bar{m}L}\right), \tag{17}$$

$$\eta = 1 + \frac{P_o + qL}{GA_s} - \frac{qL}{GA_s}\left(\frac{S_b}{(kL)^2 GA_s/L} - \frac{M_b}{\bar{m}L}\right). \tag{18}$$

Replacing the values of M_A and M_B into Eq. (4) and making $x = L$, the characteristic equation for the non-dimensional eigenvalue kL [or the critical loads $(q)_{\mathrm{cr}}$ or $(P_o)_{\mathrm{cr}}$] and the expression for the corresponding modal shape are obtained as

$$\xi\frac{\sin kL}{\beta kL}\left\{\left(1 + \frac{P_o}{GA_s}\right)\left(\alpha - \tan\frac{kL}{2}\right) + \frac{qL}{GA_s}\left[\alpha\left(1 - \frac{1}{kL}\tan\frac{kL}{2}\right) + \frac{1}{kL}\left(\frac{kL}{\tan kL} - 1\right)\right]\right\} + (kL)^2\left[\frac{J_a + J_b}{\bar{m}L^3} + \frac{r^2}{L^2}\right]$$
$$= 1 + \frac{\kappa_a + \kappa_b}{LGA_s} + \frac{P_o + (qL/2)}{GA_s}, \tag{19}$$

$$y(x) = \left[\frac{\xi}{\beta kL}(\alpha \sin kx + \cos kx) + \frac{qL/(GA_s)}{(kL)^2}\right] L\theta. \tag{20}$$

The free vibration analysis of shear beam-columns depends on 15 input variables: G, L, $\bar{m}$, r, P_o, q, A_s, κ_a, κ_b, S_a, S_b, M_a, M_b, J_a and J_b. However, these variables are grouped in Eq. (19) into the following nine dimensionless parameters: P_o/GA_s, qL/GA_s, $M_a/\bar{m}L$, $M_b/\bar{m}L$, $S_a/(GA_s/L)$, $S_b/(GA_s/L)$, $(\kappa_a + \kappa_b)/LGA_s$, $(J_a + J_b)/\bar{m}L^3$ and r/L.

The free vibration analysis can be carried out as follows: (1) knowing the nine dimensionless parameters just listed above, replace ξ, η, α and β using expressions in Eqs. (15)–(18) into the eigenvalue given by Eq. (19) and then solve it for the shear-wave number k; (2) calculate the natural frequencies $\omega = k\sqrt{GA_s/\bar{m}}$ and the corresponding numerical values of ξ, η, α and β using expressions in Eqs. (15)–(18) again, as well as coefficients

C_1 and C_2 from Eqs. (13) and (14), respectively; and (3) the modal shape $y(x)$ from Eq. (20) and free vibration waves $Y(x, t)$ and $\Theta(t)$ from Eqs. (5) and (6).

3. Static stability analysis

The static buckling loads of a 2D shear beam-column with generalized support conditions subjected to a constant axial load P_o and a uniform distributed axial load q (Fig. 1a) can be obtained directly from Eq. (19) by reducing the shear wave number k to zero (or $\omega = 0$), resulting in the following general equation:

$$\left(1+\frac{P_o+(qL/2)}{GA_s}\right)\left[\left(1+\frac{P_o}{GA_s}\right)\frac{\alpha}{\beta}-1\right]+\left(1+\frac{P_o}{GA_s}\right)\frac{qL}{2GA_s}+\frac{1}{3}\left(\frac{qL}{GA_s}\right)^2=\frac{\kappa_a+\kappa_b}{LGA_s}, \tag{21}$$

where

$$\alpha=\frac{\eta S_a+\xi S_b}{(\eta-\xi)GA_S-\xi S_b L}, \tag{22}$$

$$\beta=\alpha-\frac{S_a}{GA_s}, \tag{23}$$

$$\xi=1+\frac{P_o}{GA_s}, \tag{24}$$

$$\eta=\xi-\frac{S_b}{2GA_s/L}\left(\frac{qL}{GA_s}\right). \tag{25}$$

The corresponding static buckling shape is given by

$$y(x)=\left[\frac{q}{2GA_s}x^2+\frac{\xi}{\beta}(\alpha x+1)\right]\theta. \tag{26}$$

As shown in Fig. 1, P_o and q are positive in compression. Eq. (21) indicates that the static buckling loads $(P_o)_{cr}$ or $(qL)_{cr}$ depend on the following six variables: S_a, S_b, κ_a, κ_b, GA_s and L. Notice that the end rotational restraints κ_a and κ_b [see the right-hand side of Eq. (20)] work together as two springs in parallel.

Important Remarks: Note that: (1) The deflected shapes $Y(x, t)$ and $\Theta(t)$ are both multiplied (amplified) by θ (i.e., the rotation of the cross section of the member). (2) The effects of rotational inertia are neglected by many analysts in the classic dynamic analyses of beams shear beams, and shear buildings (Thomson [1], Blevins [2], Berg [3], Paz [4], Clough and Penzien [5], Chopra [6], among many others). This is based on the misconception that transverse equilibrium is not affected by the rotational inertia of the member as Eq. (1) indicates. (3) In the case of shear beams and shear buildings with $q = 0$ and $P_o = 0$, it is also erroneously assumed as the simplest beam with continuous mass that satisfies the standard one-dimensional shear-wave equation $(\partial^2 Y/\partial x^2) = (1/C_s^2)(\partial^2 Y/\partial t^2)$ (where: $C_s = \sqrt{GA_s/\bar{m}}$). The solution to this "simplified" second-order differential equation (which is similar to those of the string and the rod), it is erroneously assumed that requires the determination of only two constants according to its two end boundary conditions. (4) The classic solutions shown in the technical literature violate the dynamic bending moment equilibrium [represented by Eqs. (2) and (19)] even in beam-columns with zero rotational inertia, because the translational mass continues to make an important contribution to the angular momentum [10, 11]. (5) Eq. (19), that represents the member's dynamic equilibrium of the overturning moment, must be fulfilled in all cases except when one or both ends of the member are perfectly clamped (i.e., when κ_a or/and κ_b are infinity, and consequently $\theta = 0$) as shown by the analysis that follows.

4. Dynamic analysis of shear members based on zero sectional rotation ($\theta = 0$)

In the classic dynamic analysis of shear beams, shear beam-columns and shear buildings, it is assumed that the cross sections do not rotate (i.e., $\theta = 0$) [Thomson [1], Blevins [2], Berg [3], Paz [4], Clough and Penzien [5], Chopra [6], among many others]. By making $\theta = 0$, the analysis of shear members [Eqs. (1)–(12)] is reduced considerably, as follows:

$$\frac{\partial V}{\partial x} = \bar{m}\frac{\partial^2 Y}{\partial t^2}, \tag{27}$$

$$\frac{\partial M}{\partial x} = V + (P_o + qx)\frac{\partial Y}{\partial x}. \tag{28}$$

Knowing that: $V = A_s G\gamma$; $\gamma = \partial Y/\partial x$; and $\partial\Theta/\partial x \cong 0$; and assuming that the values of A_s, I, r, $\bar{m}$, and G remain constant along the span, the coupled governing Eqs. (27) and (28) become

$$\frac{\partial^2 y}{\partial x^2} - \frac{\bar{m}}{GA_s}\frac{\partial^2 Y}{\partial t^2} = 0, \tag{29}$$

$$M(x) = M_A + \int_0^x \left[(A_s G + P_o + qx)\frac{\partial Y}{\partial x}\right] \mathrm{d}x. \tag{30}$$

The solution to $Y(x, t)$ is of the form:

$$Y(x,t) = (C_1 \sin kx + C_2 \cos kx)\sin(\omega t). \tag{31}$$

In the classic analysis of the shear wave problem [Eq. (29)] C_1, C_2, and ω (or k) in Eq. (31) are determined from the following two boundary conditions. At $x = 0$

$$V_A = S_a Y(0,t) + M_a\frac{\partial^2 Y(0,t)}{\partial t^2} = (S_a - \omega^2 M_a)Y_a \tag{32}$$

and at $x = L$

$$V_B = -S_b Y(L,t) - M_b\frac{\partial^2 Y(L,t)}{\partial t^2} = (-S_b + \omega^2 M_b)Y_b. \tag{33}$$

The expressions for Y_a and Y_b which can be determined using Eq. (31) when substituted into Eqs. (32) and (33), and taking into account that $V = A_s G\gamma$, $\gamma = \partial Y/\partial x$ and $\partial\Theta/\partial x \cong 0$ the following two homogeneous equations in terms of coefficients C_1 and C_2 are obtained:

$$kC_1 - \frac{S_a - \omega^2 M_a}{GA_s}C_2 = 0, \tag{34}$$

$$\left(k \cos kL + \frac{S_b - \omega^2 M_b}{GA_s}\sin kL\right)C_1 - \left(k \sin kL - \frac{S_b - \omega^2 M_b}{GA_s}\cos kL\right)C_2 = 0. \tag{35}$$

The eigenvalue ω (or kL, taking into account that $\omega^2 = k^2 GA_s/\bar{m}$) is obtained making the determinant of Eqs. (34) and (35) equal to zero resulting in the following characteristic equation:

$$\frac{\frac{M_a+M_b}{\bar{m}L}(kL)^2 - \frac{S_a+S_b}{GA_s/L}}{\left[\frac{S_a}{GA_s/L} - \frac{M_a}{\bar{m}L}(kL)^2\right]\left[\frac{S_b}{GA_s/L} - \frac{M_b}{\bar{m}L}(kL)^2\right] - (kL)^2} = \frac{\tan kL}{kL}. \tag{36}$$

The modal shape can be obtained combining Eqs. (31) and (34) as

$$y(x) = C_2\left\{\left[\frac{S_a}{GA_s/L} - \frac{M_a}{\bar{m}L}(kL)^2\right]\frac{\sin kL}{kL}\sin kx + \cos kx\right\}. \tag{37}$$

Note that: (a) the eigenvalues are determined directly from the two boundary conditions of lateral shear equilibrium at the ends [Eqs. (32) and (33)] and Eq. (30) of bending moment becomes "silent" as indicated by Kausel [10]. (b) The left terms of Eqs. (34) and (35) are identical to those of Eqs. (11) and (12). (c) The eigenvalue kL and modal shape $y(x)$ are not affected either by the applied axial loads (P_o and q) nor by the rotatory inertias of all masses (J_a, J_b and $\bar{m}r^2$) in members with $\theta = 0$ as shown by Eqs. (36) and (37). However, the effects of all rotational inertias and the applied axial loads cannot be neglected if $\theta \neq 0$, as it is assumed by many analysts and in the technical literature in the dynamic analyses of beams, shear beams, and shear buildings [Thomson [1], Blevins [2], Berg [3], Paz [4], Clough and Penzien [5], Chopra [6], among many others]; and (d) while Eq. (19) captures the static stability of shear members with $\Theta \neq 0$ (as kL approaches zero), since Eqs. (11) and (12) are dependent of the applied axial loads. Eq. (36) on the contrary does not capture the stability of shear members with $\theta = 0$, since Eqs. (34)–(36) are independent of the applied axial loads.

5. Comprehensive examples and verification of proposed equations

Example 1. *Free vibration and stability analyses of classical shear beam-columns (with $\Theta \neq 0$).* Using Eqs. (19)–(21) and (26) determine the free vibration characteristics (i.e., the natural frequencies and the corresponding undamped modes of vibration) and buckling loads and corresponding shapes of the shear beam-columns of Fig. 2. Compare the results of the free–free and pinned–free beams presented by Kausel [10].

Solution:

(I) *Free–free shear beam-columns*: Making $\kappa_a = \kappa_b = S_a = S_b = 0$ in Eq. (19), the natural frequencies for free-free shear beam–columns [Fig. 2(a)] can be obtained from the following expression:

$$\xi \frac{\sin kL}{\beta kL}\left\{\left(1+\frac{P_o}{GA_s}\right)\left(\alpha - \tan\frac{kL}{2}\right) + \frac{qL}{GA_s}\left[\alpha\left(1-\frac{1}{kL}\tan\frac{kL}{2}\right)+\frac{1}{kL}\left(\frac{kL}{\tan kL}-1\right)\right]\right\} + (kL)^2\left[\frac{J_a+J_b}{\bar{m}L^3}+\frac{r^2}{L^2}\right] = 1+\frac{P_o+(qL/2)}{GA_s}. \tag{38}$$

The corresponding modal shapes can be determined from Eq. (20). The values of ξ, η, α and β are as follows:

$$\xi = 1+\frac{P_o}{GA_s}-\frac{qL}{GA_s}\frac{M_a}{\bar{m}L},$$

$$\eta = 1+\frac{P_o+qL}{GA_s}+\frac{qL}{GA_s}\frac{M_b}{\bar{m}L},$$

$$\alpha = \frac{(-\xi kL\sin kL)\bar{m}L-(\eta M_a+\xi M_b\cos kL)(kL)^2}{(\eta kL-\xi kL\cos kL)\bar{m}L+\xi M_b(kL)^2\sin kL}$$

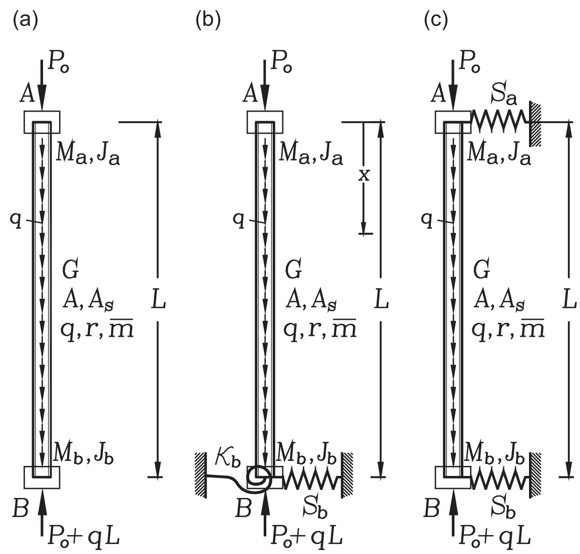


Fig. 2. Classical beam-column cases: (a) free–free; (b) cantilever; and (c) simply supported on end springs.

and

$$\beta = \alpha + \frac{M_a}{\bar{m}L} kL.$$

The corresponding static stability equation for free–free shear beam-columns is obtained from Eqs. (26) and (33) as follows:

$$\left(1 + \frac{P_o + (qL/2)}{GA_s}\right)\frac{P_o}{GA_s} + \left(1 + \frac{P_o}{GA_s}\right)\frac{qL}{2GA_s} + \frac{1}{3}\left(\frac{qL}{GA_s}\right)^2 = 0, \tag{39}$$

where $\alpha = \beta = 0$ and $\xi = \eta = 1 + (P_o/GA_s)$.

As expected, the corresponding buckling mode becomes $y(x) = \infty$. For instance, if the member has not end masses (i.e., $M_a = M_b = J_a = J_b = 0$) and is subjected to only P_o with $q = 0$, then Eqs. (38) and (20) are further reduced to

$$\left(1 + \frac{P_o}{GA_s}\right)^2 \frac{\tan(kL/2)}{(kL/2)} + (kL)^2\left(\frac{r}{L}\right)^2 = 1 + \frac{P_o}{GA_s}, \tag{40}$$

$$y(x) = \left(1 + \frac{P_o}{GA_s}\right)\frac{\theta}{k\cos(kL/2)}\sin[k(x - L/2)]. \tag{41}$$

The static buckling load [using Eq. (39) with $q = 0$] becomes $P_o = -GA_s$, that is, static buckling occurs only when the applied load P_o is in tension and equal to GA_s. Expressions (40) and (41) yield identical results to those presented by Kausel [10] and more recently by Aristizabal-Ochoa [11] in the case of free–free shear-beams.

(II) *Cantilever shear beam-columns*: Making $\kappa_a = S_a = 0$ in Eq. (19) the natural frequencies for cantilever shear beam-columns [Fig. 2(b)] can be obtained from the following expression:

$$\xi\frac{\sin kL}{\beta kL}\left\{\left(1 + \frac{P_o}{GA_s}\right)\left(\alpha - \tan\frac{kL}{2}\right) + \frac{qL}{GA_s}\left[\alpha\left(1 - \frac{1}{kL}\tan\frac{kL}{2}\right) + \frac{1}{kL}\left(\frac{kL}{\tan kL} - 1\right)\right]\right\} + (kL)^2\left[\frac{J_a + J_b}{\bar{m}L^3} + \frac{r^2}{L^2}\right]$$
$$= 1 + \frac{\kappa_b}{LGA_s} + \frac{P_o + (qL/2)}{GA_s}. \tag{42}$$

The corresponding modal shapes can be determined from Eq. (20). The values of ξ, η, α and β are as follows:

$$\xi = 1 + \frac{P_o}{GA_s} - \frac{qL}{GA_s}\frac{M_a}{\bar{m}L},$$

$$\eta = 1 + \frac{P_o + qL}{GA_s} - \frac{qL}{GA_s}\left(\frac{S_b}{(kL)^2 GA_s/L} - \frac{M_b}{\bar{m}L}\right),$$

$$\alpha = \frac{\xi\left(\frac{S_b}{GA_s/L}\cos kL - kL\sin kL\right)\bar{m}L - (\eta M_a + \xi M_b\cos kL)(kL)^2}{\left(\eta kL - \xi kL\cos kL - \frac{\xi S_b}{GA_s/L}\sin kL\right)\bar{m}L + \xi M_b(kL)^2\sin kL}$$

and

$$\beta = \alpha + \frac{M_a}{\bar{m}L} kL.$$

The corresponding static stability and deflected buckling equations for cantilever shear beam-columns were obtained from Eqs. (16) and (21) yielding the following expressions:

$$\left(1 + \frac{P_o + (qL/2)}{GA_s}\right)\left[\left(1 + \frac{P_o}{GA_s}\right) - 1\right] + \left(1 + \frac{P_o}{GA_s}\right)\frac{qL}{2GA_s} + \frac{1}{3}\left(\frac{qL}{GA_s}\right)^2 = \frac{\kappa_b}{LGA_s}, \tag{43}$$

$$y(x)=\left[\frac{q}{2GA_s}x^2+\left(1+\frac{P_o}{GA_s}\right)x-\left(1+\frac{P_o+qL/2}{GA_s}\right)L\right]\theta. \tag{44}$$

If the member is fully restrained laterally at the base [i.e., $S_b=\infty$ in Fig. 2(b)] and is subjected to both P_o and $q\neq 0$, the expressions for ξ, η, α and β for the frequency calculation are further reduced to

$$\xi=1+\frac{P_o}{GA_s}-\frac{qL}{GA_s}\frac{M_a}{\bar{m}L};\quad \eta=-\infty;\quad \alpha=-\frac{\xi\cos kL+(qL/GA_s)(M_a/\bar{m}L)}{\xi\sin kL+(qL/GA_s)(1/kL)}\quad \text{and}$$

$$\beta=\frac{\xi\cos kL\big((M_a/\bar{m}L)kL-1\big)}{\xi\sin kL+(qL/GA_s)(1/kL)}$$

For the particular case of a cantilever beam [i.e., $S_b=\infty$ and $\kappa_a=S_a=M_a=J_a=M_b=J_b=q=P_o=0$ in Fig. 2(b)], the natural frequencies and modal shapes can be obtained from Eqs. (45) and (46) as follows:

$$\frac{\tan kL}{kL}+(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{\kappa_b}{GA_sL}, \tag{45}$$

$$y(x)=\frac{\theta}{k\cos kL}\sin[k(L-x)], \tag{46}$$

where $\xi=\eta=1$ and $\alpha=\beta=-(1/\tan kL)$.

In the case of free-pinned shear beams (i.e., $\kappa_b=0$) Eqs. (45) and (46) yield identical results to those presented by by Kausel [10] and Aristizabal-Ochoa [11].

(III) *Simply supported shear beam-columns*: Making $\kappa_a=\kappa_b=0$ in Eq. (19), the natural frequency for simply supported shear beam-columns [Fig. 2(c)] can be obtained as

$$\xi\frac{\sin kL}{\beta kL}\left\{\left(1+\frac{P_o}{GA_s}\right)\left(\alpha-\tan\frac{kL}{2}\right)+\frac{qL}{GA_s}\left[\alpha\left(1-\frac{1}{kL}\tan\frac{kL}{2}\right)+\frac{1}{kL}\left(\frac{kL}{\tan kL}-1\right)\right]\right\}+(kL)^2\left[\frac{J_a+J_b}{\bar{m}L^3}+\frac{r^2}{L^2}\right]$$
$$=1+\frac{P_o+(qL/2)}{GA_s}. \tag{47}$$

The corresponding modal shapes are obtained from Eq. (20) with ξ, η, α and β given by Eqs. (15)–(18), respectively.

The static stability equation for simple supported beam-columns [Fig. 2(c)] is obtained from Eq. (21) as follows:

$$\left(1+\frac{P_o+(qL/2)}{GA_s}\right)\left[\left(1+\frac{P_o}{GA_s}\right)\frac{\alpha}{\beta}-1\right]+\left(1+\frac{P_o}{GA_s}\right)\frac{qL}{2GA_s}+\frac{1}{3}\left(\frac{qL}{GA_s}\right)^2=0. \tag{48}$$

The corresponding buckling shapes can be obtained from Eq. (26) with α, β, ξ and η given by Eqs. (22)–(25), respectively. For instance, if $P_o\neq 0$ and $q=0$, then: $\xi=\eta=1+(P_o/GA_s)$ and the static buckling loads are $(P_o)_{\text{cr}}=\big((S_aS_b)/(S_a+S_b)\big)L$ (compression) and $(P_o)_{\text{cr}}=-GA_s$ (tension). These results show that the lateral end restraints S_a and S_b work together in parallel when the member is under compression. The corresponding buckling shape under compression is $y(x)=\big(1+(P_o/GA_s)\big)\big((\alpha/\beta)x+(1/\beta)\big)\theta$, where $\alpha=-(S_a+S_b)/LS_b$ and $\beta=\alpha-S_a/(GA_s)$, and $y(x)=0$ when the member is under the tension buckling load.

On the other hand if $P_o=0$ and $q\neq 0$, Eqs. (48) and (26) are reduced to the following quadratic equations:

$$\left(\frac{S_a+S_b}{S_aS_b}\frac{4GA_s}{L}+1\right)\left(\frac{qL}{GA_s}\right)^2+6\left(\frac{S_a+S_b}{S_aS_b}\frac{GA_s}{L}-1\right)\frac{qL}{GA_s}-12=0, \tag{49}$$

$$y(x)=\left[\frac{q}{2GA_s}x^2+\frac{1}{\beta}(\alpha x+1)\right]\theta, \tag{50}$$

where

$$\alpha=\frac{[S_a/(2GA_s/L)]qL/(GA_s)-(S_a/S_b+1)}{L[1+qL/(2GA_s)]}$$

and

$$\beta = -\frac{S_a/(GA_s/L) + (S_a/S_b + 1)}{L[1 + qL/(2GA_s)]}.$$

Again, the first expression has two solutions for the critical load $(qL)_{cr}$, a compressive load and a tensile load, respectively.

For convenience, Figs. 3–5 show a series of formulas ready to use in the free vibration analysis (natural frequencies and the corresponding undamped modes of vibration) and static stability (critical loads and corresponding buckling shapes) of five different beam-columns corresponding to each of the three classical cases shown in Fig. 2 (in all cases listed $\theta \neq 0$ and the effects of the rotatory inertia $\bar{m}r^2$ are included). All formulas were obtained using Eqs. (19)–(21) and (26) as previously shown.

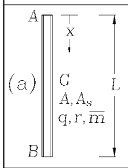
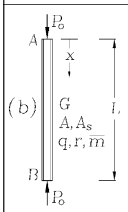
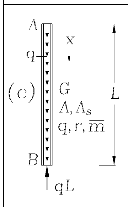
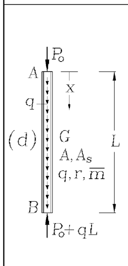
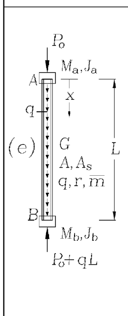
CASE	Characteristic Equation and Mode Shape	Static Stability
(a) 	$\frac{\tan(kL/2)}{(kL/2)}+(kL)^2\left(\frac{r}{L}\right)^2=1$ $y(x)=\frac{\theta}{k\cos(kL/2)}\sin[k(x-L/2)]$	N.A.
(b) 	$\left(1+\frac{P_o}{GA_s}\right)\frac{\tan(kL/2)}{(kL/2)}+(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{P_o}{GA_s}$ $y(x)=\left(1+\frac{P_o}{GA_s}\right)\frac{L\theta}{kL\cos(kL/2)}\sin[k(x-L/2)]$	$(P_o)_{cr}=0$ (compression) $(P_o)_{cr}=-GA_s$ (tension)
(c) 	$\alpha=\beta=\frac{\sin kL}{\cos kL-(1+qL/GA_s)}$ $\frac{\sin kL}{kL}\left\{1-\frac{1}{\beta}\tan\frac{kL}{2}+\frac{qL}{GA_s}\left[1-\frac{1}{kL}\tan\frac{kL}{2}+\frac{1}{\beta}\left(\frac{1}{\tan kL}+\frac{1}{kL}\right)\right]\right\}=1+\frac{qL}{2GA_s}-(kL)^2\left(\frac{r}{L}\right)^2$ $y(x)=\left[\frac{1}{kL}\left(\sin kx+\frac{1}{\beta}\cos kx\right)+\frac{qL/GA_s}{(kL)^2}\right]L\theta$	$(qL)_{cr}=0$ (compression) $(qL)_{cr}=-\frac{3}{2}GA_s$ (tension)
(d) 	$\alpha=\beta=\frac{\sin kL}{\cos kL-[1+(qL+P_o)/GA_s]}$ $\frac{\sin kL}{kL}\left(1+\frac{P_o}{GA_s}\right)\left\{\left(1+\frac{P_o}{GA_s}\right)\left(1-\frac{1}{\beta}\tan\frac{kL}{2}\right)+\frac{qL}{GA_s}\left[1-\frac{1}{kL}\tan\frac{kL}{2}+\frac{1}{\beta}\left(\frac{1}{\tan kL}+\frac{1}{kL}\right)\right]\right\}+(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{P_o+qL/2}{GA_s}$ $y(x)=\left[\frac{1+P_o/GA_s}{kL}\left(\sin kx+\frac{1}{\beta}\cos kx\right)+\frac{qL/GA_s}{(kL)^2}\right]L\theta$	$\left(1+\frac{P_o+\frac{qL}{2}}{GA_s}\right)\frac{P_o}{GA_s}+\left(1+\frac{P_o}{GA_s}\right)\frac{qL}{2GA_s}+\frac{1}{3}\left(\frac{qL}{2GA_s}\right)^2=0$ Solutions to this Quadratic Equation $(P_o)_{cr}=\frac{1}{2}\left[\pm\sqrt{(GA_s)^2-\frac{1}{3}(qL)^2}-(qL+GA_s)\right]$ or $(qL)_{cr}=\frac{3}{2}\left[\pm\sqrt{(GA_s)^2-\frac{1}{3}P_o(P_o+GA_s)}-\left(P_o+\frac{GA_s}{2}\right)\right]$
(e) 	$\xi=1+\frac{P_o}{GA_s}-\frac{qL}{GA_s}\frac{M_a}{\bar{m}L}$ $\quad\eta=1+\frac{P_o+qL}{GA_s}+\frac{qL}{GA_s}\frac{M_b}{\bar{m}L}$ $\alpha=\frac{(-\xi kL\sin kL)\bar{m}L-(\eta M_a+\xi M_b\cos kL)(kL)^2}{(\eta kL-\xi kL\cos kL)\bar{m}L+\xi M_b(kL)^2\sin kL}$; $\beta=\alpha+\frac{M_a}{\bar{m}L}kL$ $\xi\frac{\sin kL}{kL}\left(1+\frac{P_o}{GA_s}\right)\left\{\left(1+\frac{P_o}{GA_s}\right)\left(\frac{\alpha}{\beta}-\frac{1}{\beta}\tan\frac{kL}{2}\right)+\frac{qL}{GA_s}\left[\frac{\alpha}{\beta}\left(1-\frac{1}{kL}\tan\frac{kL}{2}\right)+\frac{1}{\beta kL}\left(\frac{kL}{\tan kL}-1\right)\right]\right\}+(kL)^2\left[\frac{J_a+J_b}{\bar{m}L^3}+\frac{r^2}{L^2}\right]=1+\frac{P_o+qL/2}{GA_s}$ $y(x)=\left[\frac{\xi}{\beta kL}(\alpha\sin kx+\cos kx)+\frac{qL/GA_s}{(kL)^2}\right]L\theta$	$y(x)=\infty$

Fig. 3. Free–free beam-column cases.

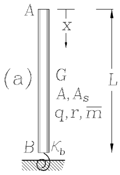
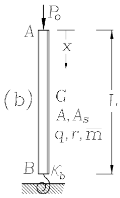
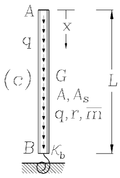
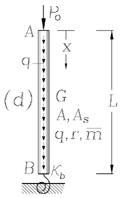
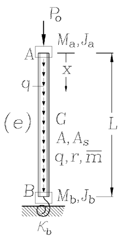
CASE	Characteristic Equation and Mode Shape	Static Stability
	$\frac{\tan kL}{kL}+(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{\kappa_b}{GA_S L}$ $y(x)=\frac{\theta}{k\cos kL}\sin[k(x-L)]$	N.A.
	$\left(1+\frac{P_o}{GA_S}\right)^2\frac{\tan kL}{kL}+(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{P_o+\kappa_b/L}{GA_S}$ $y(x)=\left(1+\frac{P_o}{GA_S}\right)\frac{L\theta}{kL\cos kL}\sin[k(x-L)]$	$\frac{P_o}{GA_S}\left(\frac{P_o}{GA_S}+1\right)-\frac{\kappa_b}{GA_S L}=0$ or $(P_o)_{cr}=\frac{GA_S}{2}\left[\pm\sqrt{1+\frac{4\kappa_b}{GA_S L}}-1\right]$ If $\kappa_b>>GA_S\,L$: $(P_o)_{cr}=\pm\sqrt{GA_S\kappa_b\,L}$ If $\kappa_b<<GA_S\,L$: $(P_o)_{cr}=0$ (compression) $(P_o)_{cr}=-GA_S$ (tension) $y(x)=\left(1+\frac{P_o}{GA_S}\right)(x-L)\theta$
	$\alpha=\beta=-\frac{kL\cos kL}{kL\sin kL+(qL/GA_S)}$ $\frac{\sin kL}{kL}\left\{1-\frac{1}{\beta}\tan\frac{kL}{2}+\frac{qL}{GA_S}\left[1-\frac{1}{kL}\tan\frac{kL}{2}+\frac{1}{\beta}\left(\frac{1}{\tan kL}-\frac{1}{kL}\right)\right]\right\}+$ $(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{qL/2+\kappa/2}{GA_S}$ $y(x)=\left[\frac{1}{kL}(\sin kx+\frac{1}{\beta}\cos kx)+\frac{qL/GA_S}{(kL)^2}\right]L\theta$	$\left(\frac{qL}{GA_S}\right)^2+\frac{3}{2}\frac{qL}{GA_S}-\frac{3\kappa_b}{GA_S L}=0$ or $(qL)_{cr}=\frac{3}{4}GA_S\left[\pm\sqrt{1+\frac{16}{3}\frac{\kappa_b}{GA_S L}}-1\right]$ If $\kappa_b>>GA_S\,L$: $(qL)_{cr}=\pm\ 3GA_S\kappa_b\ L$ If $\kappa_b<<GA_S\,L$: $(qL)_{cr}=0$ (compression) $(qL)_{cr}=\frac{3}{2}GA_S$ (tension) $y(x)=\left(1+\frac{qL\ 2}{GA_S}\right)(x^2-L^2)\theta/L$
	$\alpha=\beta=-\frac{(1+P_o/GA_S)kL\cos kL}{(1+P_o/GA_S)kL\sin kL+(qL/GA_S)}$ $\frac{\sin kL}{kL}\left(1+\frac{P_o}{GA_S}\right)\left\{\left(1+\frac{P_o}{GA_S}\right)\left(1-\frac{1}{\beta}\tan\frac{kL}{2}\right)+\frac{qL}{GA_S}\left[1-\frac{1}{kL}\tan\frac{kL}{2}+\right.\right.$ $\left.\left.\frac{1}{\beta}\left(\frac{1}{\tan kL}-\frac{1}{kL}\right)\right]\right\}+(kL)^2\left(\frac{r}{L}\right)^2=1+\frac{P_o+qL/2+\kappa_b/L}{GA_S}$ $y(x)=\left[\frac{1+P_o/GA_S}{kL}(\sin kx+\frac{1}{\beta}\cos kx)+\frac{qL/GA_S}{(kL)^2}\right]L\theta$	$\left(\frac{P_o}{GA_S}\right)^2+\left(1+\frac{qL}{GA_S}\right)\frac{P_o}{GA_S}$ $+\left[\frac{qL}{2GA_S}+\frac{1}{3}\left(\frac{qL}{GA_S}\right)^2-\frac{\kappa_b}{GA_S L}\right]=0$ or $\frac{1}{3}\left(\frac{qL}{GA_S}\right)^2+\left(\frac{1}{2}+\frac{P_o}{GA_S}\right)\frac{qL}{GA_S}$ $+\left[\frac{P_o}{GA_S}+\left(\frac{P_o}{GA_S}\right)^2-\frac{\kappa_b}{GA_S L}\right]=0$ Solutions to these two quadratic equations: $(P_o)_{cr}=\frac{GA_S}{2}\left[\pm\sqrt{1-\frac{1}{3}\left(\frac{qL}{GA_S}\right)^2+\frac{4\kappa_b}{GA_S L}}-\left(1+\frac{qL}{GA_S}\right)\right]$ $(qL)_{cr}=$ $\frac{3}{4}GA_S\left[\pm\sqrt{1-\frac{4}{3}\frac{P_o}{GA_S}\left(1+\frac{P_o}{GA_S}\right)+\frac{16}{3}\frac{\kappa_b}{GA_S L}}-\left(1+\frac{2P_o}{GA_S}\right)\right]$ Static Buckling Shape: $y(x)=\left[\frac{qL}{2GA_S}x^2+\left(1+\frac{P_o}{GA_S}\right)x-\left(1+\frac{P_o+qL/2}{GA_S}\right)L\right]\theta$
	$\xi=1+\frac{P_o}{GA_S}-\frac{qL}{GA_S}\frac{M_a}{\bar{m}L}$ $\alpha=-\frac{\xi\cos kL+\frac{qL}{GA_S}\frac{M_a}{\bar{m}L}}{\xi\sin kL+\frac{qL}{GA_S}\frac{1}{kL}}$ $\beta=\frac{\xi\cos kL\left(\frac{M_a}{\bar{m}L}kL-1\right)}{\xi\sin kL+\frac{qL}{GA_S}\frac{1}{kL}}$ $\xi\frac{\sin kL}{kL}\left\{\left(1+\frac{P_o}{GA_S}\right)\left(\frac{\alpha}{\beta}-\frac{1}{\beta}\tan\frac{kL}{2}\right)+\frac{qL}{GA_S}\left[\frac{\alpha}{\beta}\left(1-\frac{1}{kL}\tan\frac{kL}{2}\right)+\right.\right.$ $\left.\left.\frac{1}{\beta kL}\left(\frac{kL}{\tan kL}-1\right)\right]\right\}+(kL)^2\left[\frac{J_a+J_b}{\bar{m}L}+\frac{r^2}{L^2}\right]=1+\frac{P_o+qL/2+\kappa_b/L}{GA_S}$ $y(x)=\left[\frac{\xi}{\beta kL}(\alpha\sin kx+\cos kx)+\frac{qL/GA_S}{(kL)^2}\right]L\theta$	

Fig. 4. Cantilever beam-column cases.

Example 2. *Free vibration of classical shear beams (with $\theta=0$)*. Using Eqs. (36) and (37) determine the free vibration characteristic equation (natural frequencies) and the corresponding modes of vibration of shear beams with the following end conditions: (1) free–free; (2) free–free with concentrated masses at both ends; (3) free-laterally restrained; (4) laterally restrained ends; and (5) restrained at the bottom with concentrated masses at both ends.

Solution:

(1) *Free–free shear beam*: In this case $M_a=M_b=S_a=S_b=0$. Therefore using Eq. (26): $\tan kL=0$; $kL=n\pi$ (where $n=1, 2, 3, 4,\ldots$); $\omega=n\pi$ $\omega=n\pi\sqrt{(GA_s/L)/\bar{m}L}$ or natural frequency $=(n/2L)\sqrt{(GA_s/\bar{m})}\cdot$ From Eq. (37): $y(x\ )=C_2\cos kx$. These results are identical to those reported by Blevins [2, p. 176].

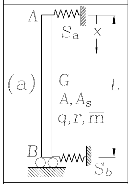
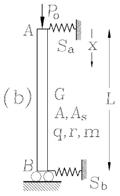
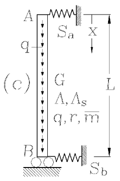
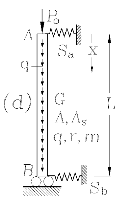
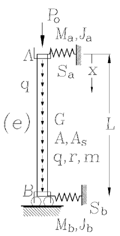
CASE	Characteristic Equation and Mode Shape	Static Stability
(a) 	$\alpha=\dfrac{\dfrac{S_a+S_b\cos kL}{GA_S}-kL\sin kL}{kL(1-\cos kL)-\dfrac{S_b}{GA_S/L}\sin kL}$ $\beta=\alpha-\dfrac{S_a}{kL(GA_S/L)}$ $\dfrac{\sin kL}{\beta kL}\left(\alpha-\tan\dfrac{kL}{2}\right)+(kL)^2\left(\dfrac{r}{L}\right)^2=1$ $y(x)=\dfrac{1}{\beta k}(\alpha\sin kx+\cos kx)$	N.A.
(b) 	$\alpha=\dfrac{\dfrac{S_a+S_b\cos kL}{GA_S}-kL\sin kL}{kL(1-\cos kL)-\dfrac{S_b}{GA_S/L}\sin kL}$ $\beta=\alpha-\dfrac{S_a}{kL(GA_S/L)}$ $\left(1+\dfrac{P_o}{GA_S}\right)^2\dfrac{\sin kL}{\beta kL}\left(\alpha-\tan\dfrac{kL}{2}\right)+(kL)^2\left(\dfrac{r}{L}\right)^2=1+\dfrac{P_o}{GA_S}$ $y(x)=\left(1+\dfrac{P_o}{GA_S}\right)\dfrac{1}{\beta k}(\alpha\sin kx+\cos kx)$	$(P_o)_{cr}=\dfrac{S_aS_b}{S_a+S_b}L$ (compression) $y(x)=\left(1+\dfrac{P_o}{GA_S}\right)\left(\dfrac{\alpha}{\beta}x+\dfrac{1}{\beta}\right)\theta$ (compression) $\alpha=-(S_a+S_b)/(S_bL)$ $\beta=-(S_a+S_b)/(S_bL)-S_a/(GA_S)$ $(P_o)_{cr}=-GA_S$ (tension) $y(x)=0$
(c) 	$\xi=1+\dfrac{qL}{GA_S}\dfrac{S_a}{(kL)^2\,GA_S/L}$ $\eta=1+\dfrac{qL}{GA_S}\left(1-\dfrac{S_b}{(kL)^2\,GA_S/L}\right)$ $\alpha=\dfrac{\dfrac{\eta S_a+\xi S_b\cos kL}{GA_S/L}-\xi kL\sin kL}{\eta kL-\xi kL\cos kL-\dfrac{\xi S_b}{GA_S/L}\sin kL}$ $\beta=\alpha-\dfrac{S_a}{(kL)GA_S/L}$ $\xi\dfrac{\sin kL}{kL}\left\{\left(1+\dfrac{P_o}{GA_S}\right)\left(\dfrac{\alpha}{\beta}-\dfrac{1}{\beta}\tan\dfrac{kL}{2}\right)+\dfrac{qL}{GA_S}\left[\dfrac{\alpha}{\beta}\left(1-\dfrac{1}{kL}\tan\dfrac{kL}{2}\right)+\dfrac{1}{\beta kL}\left(\dfrac{kL}{\tan kL}-1\right)\right]\right\}+(kL)^2\left(\dfrac{r}{L}\right)^2=1+\dfrac{qL}{2GA_S}$ $y(x)=\left[\dfrac{\xi}{\beta kL}(\alpha\sin kx+\cos kx)+\dfrac{qL/GA_S}{(kL)^2}\right]L\theta$	$(4\lambda-3)\left(\dfrac{qL}{GA_S}\right)^2+6(\lambda-2)\dfrac{qL}{GA_S}-12=0$ Solution to this Quadratic Equation: $\dfrac{(qL)_{cr}}{GA_S}=\dfrac{\pm\sqrt{\lambda(\lambda+4/3)}-\lambda+2}{4\lambda/3-1}$ $\lambda=\dfrac{S_a+S_b}{S_aS_b}\dfrac{GA_S}{L}+1$ $y(x)=\left[\dfrac{q}{2GA_S}x^2+\dfrac{1}{\beta}(\alpha x+1)\right]\theta$ $\alpha=\dfrac{[S_a/(2GA_S/L)]qL/GA_S-(S_a/S_b+1)}{L[1+qL/(2GA_S)]}$ $\beta=-\dfrac{S_a/(GA_S/L)+(S_a/S_b+1)}{L[1+qL/(2GA_S)]}$
(d) 	$\xi=1+\dfrac{P_o}{GA_S}+\dfrac{qL}{GA_S}\dfrac{S_a}{(kL)^2\,GA_S/L}$ $\eta=1+\dfrac{P_o+qL}{GA_S}-\dfrac{qL}{GA_S}\dfrac{S_b}{(kL)^2\,GA_S/L}$ $\alpha=\dfrac{\dfrac{\eta S_a+\xi S_b\cos kL}{GA_S/L}-\xi kL\sin kL}{\eta kL-\xi kL\cos kL-\dfrac{\xi S_b}{GA_S/L}\sin kL}$ $\beta=\alpha-\dfrac{S_a}{(kL)GA_S/L}$ $\xi\dfrac{\sin kL}{kL}\left\{\left(1+\dfrac{P_o}{GA_S}\right)\left(\dfrac{\alpha}{\beta}-\dfrac{1}{\beta}\tan\dfrac{kL}{2}\right)+\dfrac{qL}{GA_S}\left[\dfrac{\alpha}{\beta}\left(1-\dfrac{1}{kL}\tan\dfrac{kL}{2}\right)+\dfrac{1}{\beta kL}\left(\dfrac{kL}{\tan kL}-1\right)\right]\right\}$ $+(kL)^2\left(\dfrac{r}{L}\right)^2=1+\dfrac{P_o+qL/2}{GA_S}$ $y(x)=\left[\dfrac{\xi}{\beta kL}(\alpha\cos kx+\cos kx)+\dfrac{qL/(GA_S)}{(kL)^2}\right]L\theta$	$\left(1+\dfrac{P_o+\dfrac{qL}{2}}{GA_S}\right)\left[\left(1+\dfrac{P_o}{GA_S}\right)\dfrac{\alpha}{\beta}-1\right]+\left(1+\dfrac{P_o}{GA_S}\right)\dfrac{qL}{2GA_S}$ $+\dfrac{1}{3}\left(\dfrac{qL}{GA_S}\right)^2=0$ Where: $\alpha=\dfrac{\eta S_a+\xi S_b}{(\eta-\xi)GA_S-\xi S_bL}$; $\beta=\alpha-\dfrac{S_a}{GA_S}$ $\xi=1+\dfrac{P_o}{GA_S}$; and $\eta=\xi-\dfrac{S_b}{2GA_S}\dfrac{qL}{L\,GA_S}$ Buckling Shape: $y(x)=\left[\dfrac{q}{2GA_S}x^2+\dfrac{\xi}{\beta}(\alpha x+1)\right]\theta$
(e) 	$\xi=1+\dfrac{P_o}{GA_S}+\dfrac{qL}{GA_S}\left(\dfrac{S_a}{(kL)^2\,GA_S/L}-\dfrac{M_a}{\bar{m}L}\right)$ $\eta=1+\dfrac{P_o+qL}{GA_S}-\dfrac{qL}{GA_S}\left(\dfrac{S_b}{(kL)^2\,GA_S/L}-\dfrac{M_b}{\bar{m}L}\right)$ $\alpha=\dfrac{\left(\dfrac{\eta S_a+\xi S_b\cos kL}{GA_S/L}-\xi kL\sin kL\right)\bar{m}L-(\eta M_a+\xi M_b\cos kL)(kL)^2}{\left(\eta kL-\xi kL\cos kL-\dfrac{\xi S_b}{GA_S/L}\sin kL\right)\bar{m}L+\xi M_b(kL)^2\sin kL}$ $\beta=\alpha+\dfrac{M_a}{\bar{m}L}kL-\dfrac{S_a}{(kL)GA_S/L}$ $\xi\dfrac{\sin kL}{kL}\left\{\left(1+\dfrac{P_o}{GA_S}\right)\left(\dfrac{\alpha}{\beta}-\dfrac{1}{\beta}\tan\dfrac{kL}{2}\right)+\dfrac{qL}{GA_S}\left[\dfrac{\alpha}{\beta}\left(1-\dfrac{1}{kL}\tan\dfrac{kL}{2}\right)+\dfrac{1}{\beta kL}\left(\dfrac{kL}{\tan kL}-1\right)\right]\right\}$ $+(kL)^2\left[\dfrac{J_a+J_b}{\bar{m}L^3}+\dfrac{r^2}{L^2}\right]=1+\dfrac{P_o+qL/2}{GA_S}$ $y(x)=\left[\dfrac{\xi}{\beta kL}(\alpha\sin kx+\cos kx)+\dfrac{qL/(GA_S)}{(kL)^2}\right]L\theta$	

Fig. 5. Simply supported beam-column cases.

(2) *Free-free shear beam with masses at both ends*: In this case $S_a = S_b = 0$; therefore from Eq. (36)

$$\frac{\tan kL}{kL} = \frac{(M_a + M_b)\bar{m}L}{M_aM_b(kL)^2 - (\bar{m}L)^2}$$

and from Eq. (37):

$$y(x) = C_2\left[-\frac{M_a}{\bar{m}L}kL\sin kx + \cos kx\right].$$

When $M_a = 0$, the eigenvalue equation is reduced to $(\tan kL/kL) = -(M_b/\bar{m}L)$ and the modal shape to $y(x) = C_2\cos kx$. These results are identical to those reported by Blevins [2, p. 178].

(3) *Free-laterally restrained shear beam*: In this case $M_a = M_b = S_a = 0$. Therefore using Eq. (36) $kL\tan kL = (S_bL/GA_s)$. From Eq. (37): $y(x) = C_2\cos kx$. Note that if $M_a = M_b = S_b = 0$, the results are $kL\tan kL = (S_aL/GA_s)$ and $y(x) = (C_2/\tan kL)[\sin kx + \cos kx/\tan kL]$. These last two results are identical to those reported by Blevins [2, p. 177].

(4) *Shear Beam with Laterally Restrained Ends*: In this case $M_a = M_b = 0$. Therefore using Eq. (36)

$$\frac{\tan kL}{kL} = -\frac{(S_a + S_b)(GA_s/L)}{S_aS_b - (kL)^2(GA_s/L)^2}$$

and from Eq. (37): $y(x) = C_2[(S_a/GA_sk)\sin kx + \cos kx]$. Note that if $S_b = \infty$, the results are $(\tan kL/kL) = -(GA_s/L)/S_a$ and $y(x) = C_1\sin kx$. These last two results are identical to those reported by Blevins [2, p. 177].

(5) *Beam laterally restrained at the bottom and with end masses at both ends*: In this case $S_a = 0$. Therefore using Eq. (36):

$$\frac{\tan kL}{kL} = -\frac{S_b/(GA_s/L) - \big((M_a + M_b)/\bar{m}L\big)(kL)^2}{(M_a/\bar{m}L)(kL)^2\left[S_b/(GA_s/L) - (M_b/\bar{m}L)(kL)^2\right] - (kL)^2}$$

and from Eq. (37): $y(x) = (C_2/\sin kL)\sin k(L - x)$. These two results are identical to those reported by Blevins [2, p. 177].

Example 3. *Effects of self-weight and base rotational stiffness on the free vibration and lateral stability of a high-rise shear building*. Using the expressions for a cantilever beam-column including its own weight q [listed as case (c) in Fig. 4], calculate the natural frequencies and the corresponding modal shapes of the first three modes of the shear building shown in Fig. 6f. Assume the following properties: $GA_s = 1/3 \times 10^6$ Kips (1.48273×10^6 kN); $r^2 = 75\,\text{ft}^2$ ($6.9677\,\text{m}^2$); $\bar{m} = 2\,\text{Kips} \times \text{s}^2/\text{ft}^2$ (9757 kg/m); $q = 64.4$ Kip/ft (95,761 kN/m); and story height = 12 ft (3.658 m). Study also the effects of the building height assuming the following values: $L = 60$, 120, 240, 360 and 480 ft (18.29, 36.56, 73.15, 109.73, and 146.30 m) corresponding to 5, 10, 20, 30 and 40-story shear buildings, respectively. Compare the results with those suggested by Clough and Penzien [5, p. 663] for a completely clamped base for the following building's base rotational stiffness: $\kappa_b = 1$, 10, 100, 1000 times κ_{cr} [where κ_{cr} is the required minimum rotational stiffness at the base obtained making the corresponding critical buckling load $(qL)_{cr}$ equal to its total self-weight of the building qL].

Solution: The required minimum rotational stiffness at the base is obtained from Eq. (36) with $P_o = 0$ or from the expression for the static stability listed in Fig. 4 case (c) as follows:

$$\kappa_{cr} = \frac{GA_sL}{3}\left[\left(\frac{qL}{GA_s}\right)^2 + \frac{3}{2}\frac{qL}{GA_s}\right]. \tag{51}$$

In Tables 1–5 are the values of κ_{cr} for the five different story heights along with the corresponding calculated kL values for the first three modes of vibration which were obtained using the following expression:

$$\frac{\sin kL}{kL}\left\{1 - \frac{1}{\beta}\tan\frac{kL}{2} + \frac{qL}{GA_s}\left[1 - \frac{1}{kL}\tan\frac{kL}{2} + \frac{1}{\beta}\left(\frac{1}{\tan kL} - \frac{1}{kL}\right)\right]\right\} + (kL)^2\left(\frac{r}{L}\right)^2 = 1 + \frac{qL/2 + \kappa_b/L}{GA_s}, \tag{52}$$

where

$$\beta = -\frac{kL\cos kL}{kL\sin kL + (qL/GA_s)}. \tag{53}$$

In Tables 1–5 are also (in the last columns) the compressive static critical loads $(qL)_{cr}$ for the different values of κ_b and story heights using the expression:

$$(qL)_{cr} = \frac{3}{4}GA_s\left[\sqrt{1 + \frac{16}{3}\frac{\kappa_b}{GA_sL}} - 1\right].$$

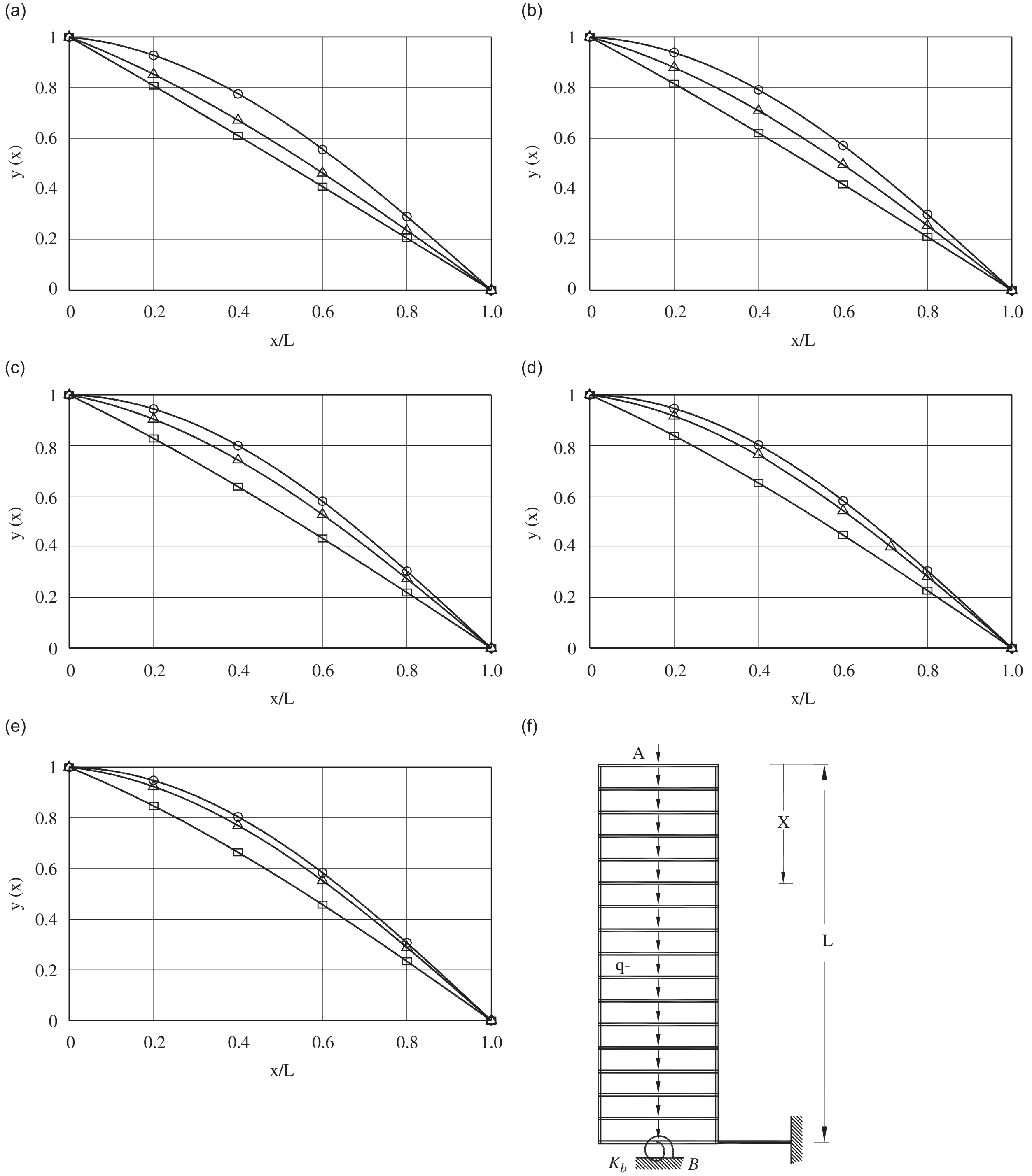


Fig. 6. Example 3: Effects of shear-building height and κ_b on the first-mode shape: (a) 5-story; (b) 10-story; (c) 20-story; (d) 30-story; (e) 40-story; and (f) mechanical model. —○—, $\kappa_b = 1000\kappa_{cr}$; —△—, $\kappa_b = 100\kappa_{cr}$; —□—, $\kappa_b = 10\kappa_{cr}$.

Table 1
kL-values and buckling loads for 5-story shear building

κ_b	1st-mode	2nd-mode	3rd-mode	$(qL)_{cr}/GA_s$
$1 \times \kappa_{cr}$	0	4.3583	6.7229	0.0115920
$10 \times \kappa_{cr}$	0.3725	4.3837	6.8366	0.1089085
$100 \times \kappa_{cr}$	0.9957	4.5258	7.5376	0.7714261
$1000 \times \kappa_{cr}$	1.4697	4.6791	7.8310	3.5026315

$L = 60\,\text{ft}$, $\kappa_{cr} = 0.005840791 \times GA_sL = 116{,}815.83\,\text{Kip-ft/Rad}$.

Table 2
kL-values and buckling loads for 10-story shear building

κ_b	1st-mode	2nd-mode	3rd-mode	$(qL)_{cr}/GA_s$
$1 \times \kappa_{cr}$	0	4.477	7.6732	0.0231840
$10 \times \kappa_{cr}$	0.5211	4.5013	7.6962	0.2068882
$100 \times \kappa_{cr}$	1.1958	4.6095	7.7851	1.2733264
$1000 \times \kappa_{cr}$	1.5177	4.6957	7.8437	5.2396576

$L = 120\,\text{ft}$, $\kappa_{cr} = 0.011771166 \times GA_sL = 470{,}846.64\,\text{Kip-ft/Rad}$.

Table 3
kL-values and buckling loads for 20-story shear building

κ_b	1st-mode	2nd-mode	3rd-mode	$(qL)_{cr}/GA_s$
$1 \times \kappa_{cr}$	0	4.5035	7.7198	0.0463680
$10 \times \kappa_{cr}$	0.6988	4.5401	7.7442	0.3811587
$100 \times \kappa_{cr}$	1.3445	4.6503	7.8152	2.0307731
$1000 \times \kappa_{cr}$	1.5433	4.7124	7.8540	7.7508524

$L = 240\,\text{ft}$, $\kappa_{cr} = 0.023900664 \times GA_sL = 1{,}912{,}053.10\,\text{Kip-ft/Rad}$.

Table 4
kL-values and buckling loads for 30-story shear building

κ_b	1st-mode	2nd-mode	3rd-mode	$(qL)_{cr}/GA_s$
$1 \times \kappa_{cr}$	0	4.5138	7.7278	0.0695520
$10 \times \kappa_{cr}$	0.8134	4.5610	7.7582	0.5361395
$100 \times \kappa_{cr}$	1.4071	4.6679	7.8260	2.6380744
$1000 \times \kappa_{cr}$	1.5520	4.7124	7.8540	9.7251124

$L = 360\,\text{ft}$, $\kappa_{cr} = 0.036388494 \times GA_sL = 4{,}366{,}619.23\,\text{Kip-ft/Rad}$.

Table 5
kL-values and buckling loads for 40-story shear building

κ_b	1st-mode	2nd-mode	3rd-mode	$(qL)_{cr}/GA_s$
$1 \times \kappa_{cr}$	0	4.5217	7.7318	0.0927360
$10 \times \kappa_{cr}$	0.8963	4.5772	7.7676	0.6781245
$100 \times \kappa_{cr}$	1.4418	4.6780	7.8260	3.1657243
$1000 \times \kappa_{cr}$	1.5708	4.7124	7.8540	11.4264718

$L = 480\,\text{ft}$, $\kappa_{cr} = 0.049234655 \times GA_sL = 7{,}877{,}544.84\,\text{Kip-ft/Rad}$.

The tensile buckling loads can be calculated directly adding $1.5GA_s$ to the compressive values listed in Tables 1–5 (For instance, in Table 1 for the 5-story shear building the tensile buckling values would be 1.5011592, 1.5115126, 1.6082042, 2.2674465 times GA_s, for $\kappa_b = 1$, 10, 100, 1000 times κ_{cr}, respectively).

Figs. 6–8 show the effects of the number of stories and the value of κ_b on the first three modal shapes which are given by the expression:

$$y(x) = \left[\frac{1}{kL}(\sin kx + \frac{1}{\beta}\cos kx) + \frac{qL/(GA_s)}{(kL)^2}\right]L\theta.$$

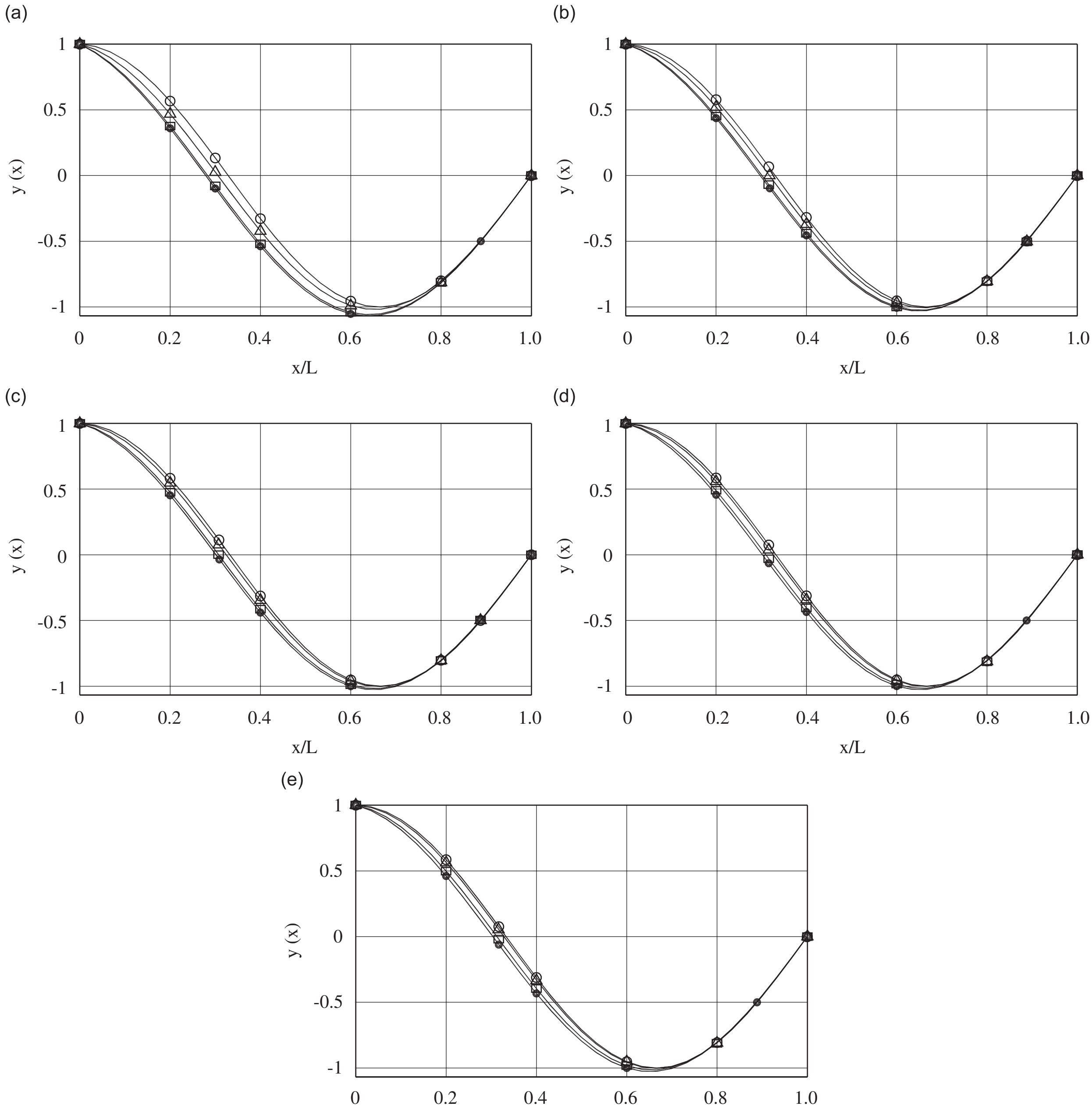


Fig. 7. Example 3: Effects of building height and κ_b on the second-mode shape: (a) 5-story; (b) 10-story; (c) 20-story; (d) 30-story; and (e) 40-story. —○—, $\kappa_b = 1000\kappa_{cr}$; —△—, $\kappa_b = 100\kappa_{cr}$; —□—, $\kappa_b = 10\kappa_{cr}$; —●—, $\kappa_b = \kappa_{cr}$.

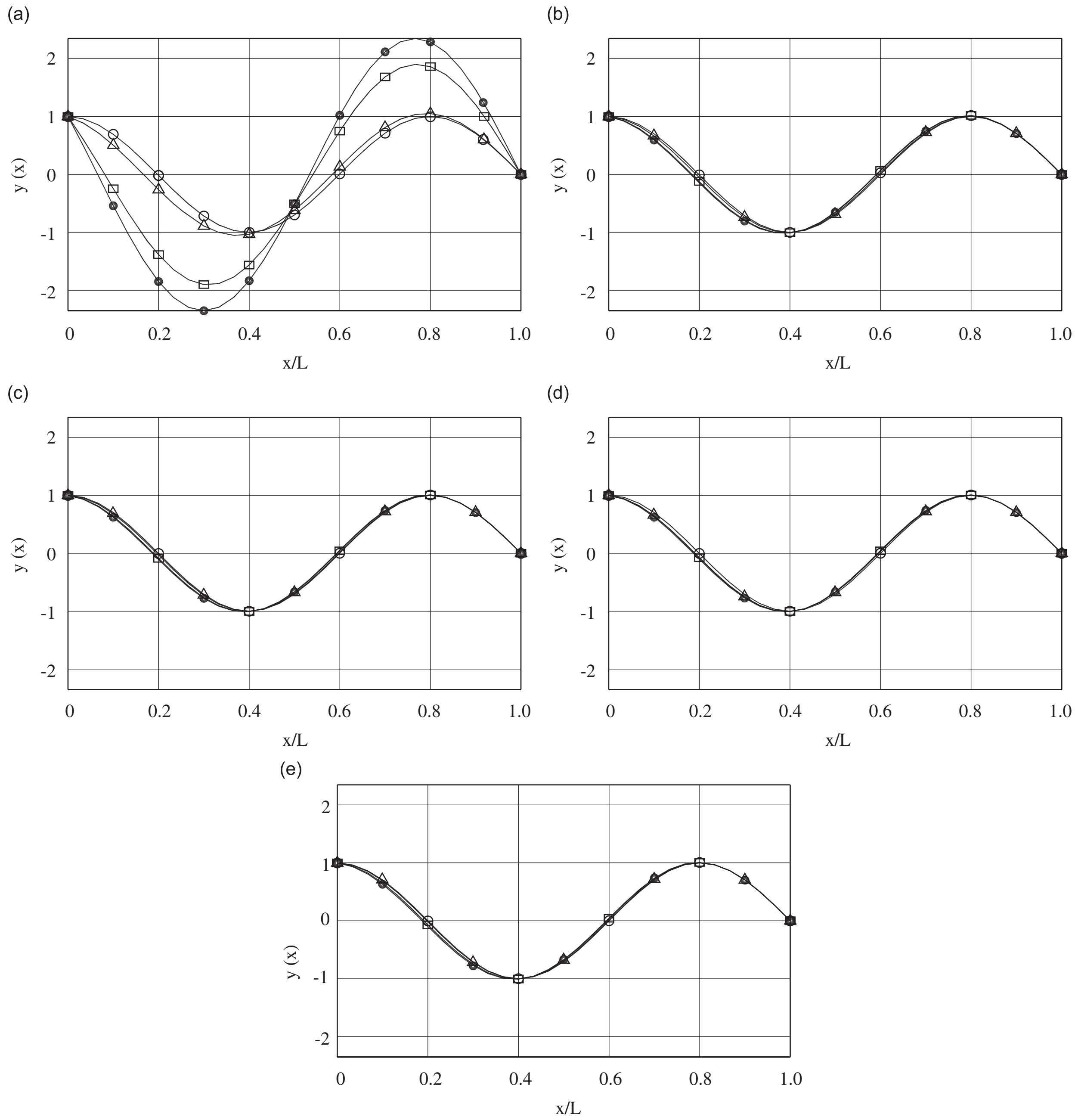


Fig. 8. Example 3: Effects of shear-building height and κ_b on the 3rd-mode shape: (a) 5-story; (b) 10-story; (c) 20-story; (d) 30-story; and (e) 40-story. —○—, $\kappa_b = 1000\kappa_{cr}$; —△—, $\kappa_b = 100\kappa_{cr}$; —□—, $\kappa_b = 10\kappa_{cr}$; —●—, $\kappa_b = \kappa_{cr}$.

Notce that the modal shapes were normalized making the value of $y(0)$ equal to 1.The critical buckling shapes are all parabolas given by $y(x) = \left[1 + \left((qL)_{cr}/2GA_s\right)\right](x^2 - L^2)\theta/L$.

Note that: (1) the first-mode of vibration (natural frequency and modal shape) of all five cases is much more sensitive to the degree of rotational restraint at the base (κ_b) than the higher modes; (2) the natural frequencies and modal shapes of the 5- and 10-story shear buildings are more sensitive to the degree of rotational restraint at the base (κ_b); (3) when this value is equal or lower than the critical value (i.e., $\kappa_b \leqslant \kappa_{cr}$), the shear building buckles laterally with the lateral deflection becoming infinity and the value of the fundamental frequency becoming zero or imaginary; (4) the values of kL tend to those indicated in the technical literature (i.e., $kL = \pi/2$, $3\pi/2$, $5\pi/2$, etc. for shear buildings completely clamped at the base [5, p. 664] as the value of κ_b increases, for instance when $\kappa_b > 1000\kappa_{cr}$ it can be assumed that the 40-story shear building is clamped at the base; (5) the shape of the first-mode of vibration is practically a straight line for values of $\kappa_b < 10\kappa_{cr}$ and it approaches a sine wave [5, p. 664] for values $\kappa_b > 1000\kappa_{cr}$.

6. Summary and conclusions

The stability and dynamic analyses (i.e., the critical loads, buckling modes, natural frequencies and corresponding modes of vibration) of a 2D shear beam-column with generalized boundary conditions (i.e., with semi-rigid flexural restraints and lateral bracings as well as lumped masses at both ends) subjected to linearly distributed axial load along its span are presented in a classical manner. The proposed model includes the simultaneous effects (or couplings) of shear deformations, translational and rotational inertias of all masses considered, the linearly applied axial load along the span, and the end restraints (rotational and lateral bracings at both ends). The proposed model shows that the lateral stability and dynamic behavior of 2D shear beam-columns are highly sensitive to the coupling effects just mentioned, particularly in members with limited end rotational restraints and lateral bracings. The proposed method allows the analyst to study the stability and free vibration behavior of 2D shear-beams, shear beam-columns and shear-buildings under any end conditions.

It is shown that the free vibration analysis of shear beam-columns depends on 15 input variables G, L, $\bar{m}$, r, P_o, q, A_s, κ_a, κ_b, S_a, S_b, M_a, M_b, J_a and J_b. However, these variables can be grouped into nine dimensionless parameters: P_o/GA_s, qL/GA_s, $M_a/\bar{m}L$, $M_b/\bar{m}L$, $S_a/(GA_s/L)$, $S_b/(GA_s/L)$, $(\kappa_a+\kappa_b)/LGA_s$, $(J_a+J_b)/\bar{m}L^3$and r/L. The proposed equations reproduce, as special cases: (1) the non-classical vibration modes of shear beam-columns including the inversion of modes of vibration (i.e. higher modes crossing lower modes) in shear beam-columns with soft end conditions, and the phenomena of double frequencies at certain values of beam slenderness (L/r) and (2) the phenomena of tension buckling in shear beam-columns. These phenomena have been discussed recently by Aristizabal-Ochoa [11] and Kelly [13].

It is shown that for shear beams, shear beam-columns and shear buildings that are not perfectly clamped against rotation of its cross section: (i) the deflected shapes $Y(x, t)$ and $\Theta(x, t)$ are both multiplied (amplified) by the rotation of the member cross section θ; (ii) the effects of rotational inertia are generally neglected by most analysts in the classical dynamic analyses of beams, shear beams, and shear buildings violating the principle of conservation of angular momentum as shown by Kausel [10]. This is based on the misconception that the transverse equilibrium is not affected by the rotational inertia of the member; (iii) in the particular case of a shear beam with no axial load applied ($q = 0$ and $P_o = 0$), it is also erroneously assumed by most analysts as the simplest beam with continuous mass that satisfies the standard one-dimensional shear wave equation $(\partial^2 Y/\partial x^2) = (1/C_s^2)(\partial^2 Y/\partial t^2)$. In the solution to this "simplified" second-order differential equation is assumed that it only requires the determination of two constants according to its two end boundary conditions (solution that is identical to that of a string); (iv) the classical solutions shown in the technical literature violate the principle of conservation of angular momentum even in beam-columns with zero rotational inertia, because the translational mass continues to make an important contribution to the angular momentum [10, 11]; and (v) Eq. (19), that represents the member's dynamic equilibrium of the overturning moment, must be fulfilled in all cases except when one or both ends of the member are perfectly clamped (i.e., when κ_a or/and κ_b are infinity and consequently $\Theta = \theta = 0$) as shown by the author herein.

It is also shown that for shear beams, shear beam-columns and shear buildings that are perfectly clamped against rotation of its cross section: (a) the eigenvalues are determined directly from the two boundary conditions for lateral shear equilibrium at the ends Eqs. (32) and (33) and the bending moment Eq. (30) becomes "silent" as indicated by Kausel [10]; (b) the left terms of Eqs. (34) and (35) are identical to those of Eqs. (11) and (12); (c) the eigenvalue kL and modal shape $y(x)$ are not affected by the applied axial loads (P_o and q) or by the rotatory inertias of all masses (J_a, J_b, and $\bar{m}r^2$) in members with $\theta = 0$ as shown by Eqs. (36) and (37). However, the effects of all rotational inertias and the applied axial loads can not be neglected when $\theta \neq 0$, as it is wrongly assumed by many analysts and in the technical literature in the dynamic analyses of beams, shear beams, and shear buildings; and (d) while the proposed eigenvalue expression [Eq. (19)] captures the static stability of shear members with $\theta \neq 0$ (as kL approaches zero), since Eqs. (11) and (12) are dependent of the applied axial loads. However, the static stability Eq. (36), on the contrary, does not capture the stability of shear members with $\theta = 0$ since Eqs. (34)–(36) are independent of the applied axial loads.

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References

[1] W.T. Thomson, *Theory of Vibration with Applications*, Prentice-Hall, Englewood Cliffs, NJ, 1972, pp. 275–276 (Chapter 8).
[2] R.D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold Co, New York, 1979, pp. 171–176 (Chapter 8).
[3] G.V. Berg, *Elements of Structural Dynamics*, Prentice-Hall, Englewood Cliffs, NJ, 1989.
[4] M. Paz, *Structural Dynamics: Theory and Computation*, 3rd ed, Vn Nostrand Reinhold Co., New York, NY, 1991, pp. 201–296 (Chapters 9–13).
[5] R.W. Clough, J. Penzien, *Dynamics of Structures*, second ed, McGraw–Hill Book Co., New York, NY, 1993, p. 628 (Chapter 26).
[6] A. Chopra, *Structural Dynamics*, seocond ed, McGraw–Hill Book Co., New York, NY, 1998, p. 628 (Chapter 26).
[7] A.N. Kounadis, Stability of elastically restrained Timoshenko cantilevers with attached masses subjected to a follower force, *ASME Transactions, Series E-Journal of Applied Mechanics* 44 (December) (1977) 731–736.
[8] A.N. Kounadis, On the derivation of equations of motion for a vibrating Timoshenko column, *Journal of Sound and Vibration* 73 (November) (1980) 177–184.
[9] A.N. Kounadis, D. Sophianopoulos, The effect of axial inertia on the bending eigenfrequencies of a Timoshenko two-bar frame, *Earthquake Engineering and Structural Dynamics* 14 (1986) 429–437.
[10] E. Kausel, Nonclassical modes of unrestrained shear beams, *ASCE Journal of Engineering Mechanics* 128 (June) (2002) 663–667.
[11] J. Dario Aristizabal-Ochoa, Timoshenko beam-column with generalized end conditions and nonclassical modes of vibration of shear beams, *ASCE Journal of Engineering Mechanics* 130 (October) (2004) 1152–1159.
[12] S. Timoshenko, J. Gere, *Theory of Elastic Stability*, second ed, McGraw-Hill, New York, 1961, p. 134.
[13] Aristizabal-Ochoa, J. Dario, *ASCE Journal of Engineering Mechanics* 131 (January) (2005) 106–108.